CHEBYSHEV UPPER ESTIMATES FOR BEURLING'S GENERALIZED PRIME NUMBERS

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Abstract. Let N be the counting function of a Beurling generalized number system and let π be the counting function of its primes. We show that the L^1-condition

$$\int_1^\infty \left|\frac{N(x)-ax}{x}\right| \frac{\mathrm{d}x}{x} < \infty$$

and the asymptotic behavior

$$N(x) = ax + O\left(\frac{x}{\log x}\right) ,$$

for some $a > 0$, suffice for a Chebyshev upper estimate

$$\frac{\pi(x)\log x}{x} \leq B < \infty .$$

1. Introduction

Let $P = \{p_k\}_{k=1}^{\infty}$ be a set of Beurling generalized primes, namely, a non-decreasing sequence of real numbers $1 < p_1 \leq p_2 \leq \cdots \leq p_k \to \infty$. The sequence $\{n_k\}_{k=1}^{\infty}$ denotes its associated set of generalized integers [2, 3]. Consider the counting functions of generalized integers and primes

$$N(x) = N_P(x) = \sum_{n_k < x} 1 \quad \text{and} \quad \pi(x) = \pi_P(x) = \sum_{p_k < x} 1 .$$

Beurling's problem consists in finding mild conditions over N that ensure a certain asymptotic behavior for π. This problem has been extensively investigated in connection with the prime number theorem (PNT), i.e.,

$$\pi(x) \sim \frac{x}{\log x} , \quad x \to \infty , \tag{1}$$

and Chebyshev two-sided estimates, that is,

$$0 < \liminf_{x\to\infty} \frac{\pi(x)\log x}{x} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\pi(x)\log x}{x} < \infty . \tag{2}$$

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On the other hand, there are no mild hypotheses in the literature for Chebyshev upper estimates,

$$\limsup_{x\to\infty} \frac{\pi(x)\log x}{x} < \infty \,. \tag{3}$$

The purpose of this article is to study asymptotic requirements over N that imply the Chebyshev upper estimate (3).

Beurling [3] proved that

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) , \quad x \to \infty \quad (a > 0) \,, \tag{4}$$

where $\gamma > 3/2$, suffices for the PNT (1) to hold. See [3, 10, 13] for more general PNT. Beurling's condition is sharp, because when $\gamma = 3/2$ there are generalized number systems for which the PNT fails [3, 5]. For $\gamma < 1$, not even Chebyshev estimates need to hold, as follows from an example of Hall [9] (see also [1]). Diamond has shown [6] that (4) with $\gamma > 1$ is enough to obtain Chebyshev two-sided estimates (2). Furthermore, he conjectured [7] that the weaker hypothesis

$$\int_{1}^{\infty} \left|\frac{N(x) - ax}{x}\right| \frac{\mathrm{d}x}{x} < \infty \,, \quad \text{with } a > 0 \,, \tag{5}$$

would be enough for (2). His conjecture was shown to be false by Kahane [11]. Nevertheless, the author has recently shown [15] that if one adds to (5) the condition

$$N(x) = ax + o\left(\frac{x}{\log x}\right) , \quad x \to \infty \,, \tag{6}$$

then (2) is fulfilled, extending thus earlier results from [6, 18].

It is natural to replace the little o symbol in (6) by an O growth estimate and investigate the effect of this new condition on the asymptotic distribution of the generalized primes. It turns out that one gets a Chebyshev upper estimate in this case. Our main goal is to give a proof of the following theorem.

Theorem 1. *Diamond's L^1-condition* (5) *and the asymptotic behavior*

$$N(x) = ax + O\left(\frac{x}{\log x}\right) , \quad x \to \infty \,, \tag{7}$$

suffice for the Chebyshev upper estimate (3).

2. Notation

We will give an analytic proof of Theorem 1. Our technique follows distributional ideas already used in [13, 15, 16]. It employs the Wiener division theorem [12, Chap. 2] and the operational calculus for the Laplace transform of Schwartz distributions [4, 17]. The Schwartz spaces of test functions and distributions are denoted as $\mathcal{D}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, see [8, 14, 17] for

their properties. If $f \in \mathcal{S}'(\mathbb{R})$ has support in $[0,\infty)$, its Laplace transform is well defined as

$$\mathcal{L}\{f;s\} = \left\langle f(u), e^{-su}\right\rangle , \quad \Re e\, s > 0 ,$$

and the Fourier transform $\hat{f}$ is the distributional boundary value [4] of $\mathcal{L}\{f;s\}$ on $\Re e\, s = 0$. We use the notation H for the Heaviside function, it is simply the characteristic function of $(0,\infty)$.

Observe that (3) is equivalent to

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} < \infty , \tag{8}$$

where ψ is the Chebyshev function

$$\psi(x) = \psi_P(x) = \sum_{n_k<x} \Lambda(n_k) ,$$

as follows from [2, Lem. 2E].

3. Proof of Theorem 1

Assume (5) and (7). Set $T(u) = e^{-u}\psi(e^u)$. We must show (8), that is,

$$\limsup_{u\to\infty} T(u) < \infty . \tag{9}$$

The crude inequality $T(u) \le ue^{-u}N(e^u) = O(u)$ implies that $T \in \mathcal{S}'(\mathbb{R})$. The proof of (9) depends upon estimates on convolution averages of T:

Lemma 1. *There exists $c > 0$ such that*

$$\int_{-\infty}^{\infty} T(u)\hat{\phi}(u-h)\mathrm{d}u = O(1) , \tag{10}$$

whenever $\phi \in \mathcal{D}(-c,c)$.

Indeed, suppose that Lemma 1 has been already established. Choose then in (10) a test function $\phi \in \mathcal{D}(-c,c)$ such that $\hat{\phi}$ is non-negative. Since $\psi(e^u)$ is non-decreasing, we have $e^{-u}T(h) \le T(u+h)$ whenever u and h are positive. Setting $C = \int_0^\infty e^{-u}\hat{\phi}(u)\mathrm{d}u > 0$, we obtain that

$$T(h) \le \frac{1}{C}\int_0^\infty T(u+h)\hat{\phi}(u)\mathrm{d}u = O(1) ,$$

and Theorem 1 follows at once. It remains to prove the lemma.

Proof of Lemma 1. Set $E_1(u) := e^{-u}N(e^u) - aH(u)$ and $E_2(u) = uE_1(u)$. The assumptions (5) and (7) take the form $E_1 \in L^1(\mathbb{R})$ and $E_2 \in L^\infty(\mathbb{R})$. Consider

$$G(s) = \zeta(s) - \frac{a}{s-1} = s\mathcal{L}\{E_1; s-1\} + a .$$

Taking $\Re e\, s \to 1^+$, in the distributional sense, we obtain $G(1+it) = (1+it)\hat{E}_1(t)+a$. Since $E_1 \in L^1(\mathbb{R})$, $\hat{E}_1$ is continuous; therefore $G(s)$ extends to a continuous function on $\Re e\, s = 1$. Consequently, $(s-1)\zeta(s)$ is continuous on

$\Re e\, s = 1$ and there exists $c > 0$ such that $it\zeta(1+it) \neq 0$ for all $t \in (-3c, 3c)$. Next, we study the boundary values, on the line segment $1 + i(-c, c)$, of

$$\mathcal{L}\left\{T(u); s-1\right\} = \mathcal{L}\left\{\psi(e^u); s\right\} = -\frac{\zeta'(s)}{s\zeta(s)}\,.$$

A quick calculation shows that

$$-\frac{\zeta'(s)}{s\zeta(s)} = \frac{\mathcal{L}\left\{E_2'; s-1\right\}}{(s-1)\zeta(s)} - \frac{(2s-1)\mathcal{L}\left\{E_1; s-1\right\} + a}{s(s-1)\zeta(s)} - \frac{1}{s} + \frac{1}{s-1}\,, \tag{11}$$

Consider the boundary distributions

$$g_1(t) = \lim_{\sigma \to 1^+} \frac{\mathcal{L}\left\{E_2'; \sigma - 1 + it\right\}}{(\sigma - 1 + it)\zeta(\sigma + it)} \quad \text{in } \mathcal{S}'(\mathbb{R})\,,$$

and

$$g_2(t) = -\lim_{\sigma \to 1^+} \left(\frac{(2\sigma - 1 + 2it)\mathcal{L}\left\{E_1; \sigma - 1 + it\right\} + a}{(\sigma + it)(\sigma - 1 + it)\zeta(\sigma + it)} + \frac{1}{\sigma + it} \right) \quad \text{in } \mathcal{S}'(\mathbb{R})\,.$$

Taking boundary values in (11), we have $\hat{T}(t) = g_1(t) + g_2(t) + \hat{H}(t)$, where H is the Heaviside function. Fix $\phi \in \mathcal{D}(-c, c)$. Notice that g_2 is actually a continuous function on $(-3c, 3c)$, thus,

$$\begin{aligned}\int_{-\infty}^{\infty} T(u)\hat{\phi}(u-h)\mathrm{d}u &= \left\langle g_1(t), e^{iht}\phi(t) \right\rangle + \int_{-c}^{c} e^{iht} g_2(t)\phi(t)\mathrm{d}t + \int_{-h}^{\infty} \hat{\phi}(u)\mathrm{d}u \\ &= \left\langle g_1(t), e^{iht}\phi(t) \right\rangle + o(1) + O(1)\,.\end{aligned}$$

Our task is then to demonstrate that $\left\langle g_1(t), e^{iht}\phi(t) \right\rangle = O(1)$. Let $M \in \mathcal{S}'(\mathbb{R})$ be the distribution supported in the interval $[0, \infty)$ that satisfies $\mathcal{L}\left\{M; s-1\right\} = ((s-1)\zeta(s))^{-1}$. Notice that $g_1 = (\widehat{E_2' * M})$. Fix an even function $\eta \in \mathcal{D}(-3c, 3c)$ such that $\eta(t) = 1$ for all $t \in (-2c, 2c)$. Then, $\eta(t)it\zeta(1+it) \neq 0$ for all $t \in (-2c, 2c)$; moreover, it is the Fourier transform of the L^1-function $\chi_1 * E_1 + \chi_2$, where $\hat{\chi}_1(t) = it(1+it)\eta(t)$ and $\hat{\chi}_2(t) = a(1+it)\eta(t)$. We can therefore apply the Wiener division theorem [12, p. 88] to $\eta(t)it\zeta(1+it)$ and $\phi(t)$. So we find $f \in L^1(\mathbb{R})$ such that

$$\hat{f}(t) = \frac{\phi(t)}{\eta(t)it\zeta(1+it)}\,.$$

Hence,

$$\left\langle g_1(t), e^{iht}\phi(t) \right\rangle = \left\langle (E_2' * M)(u), \hat{\phi}(u-h) \right\rangle = (E_2 * (\hat{\eta})' * f)(h) = O(1)\,,$$

because $E_2 \in L^{\infty}(\mathbb{R})$ and $(\hat{\eta})' * f \in L^1(\mathbb{R})$, whence (10) follows.

□

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